Lie bialgebra structures on extended Schrödinger-Virasoro Lie algebra[1]

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Abstract. In this paper, Lie bialgebra structures on the extended Schrödinger-Virasoro Lie algebra $\mathcal{L}$ are classified. It is obtained that all the Lie bialgebra structures on $\mathcal{L}$ are triangular coboundary. As a by-product, it is derived that the first cohomology group $H^1(\mathcal{L}, \mathcal{L} \otimes \mathcal{L})$ is trivial.

Key words: Lie bialgebras, Yang-Baxter equation, Extended Schrödinger-Virasoro Lie algebras.

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§1. Introduction

The Schrödinger-Virasoro Lie algebras [6] were introduced in the context of non-equilibrium statistical physics during the process of investigating the free Schrödinger equations. They are closely related to the Schrödinger algebra and the Virasoro algebra, both of which play important roles in many areas of mathematics and physics (e.g., statistical physics, integrable system) and have been investigated in a series of papers [7–10, 13, 18, 21, 25]. In order to investigate vertex representations of the Schrödinger-Virasoro Lie algebra, J. Unterberger introducted (see Definition 1.5 in [22]) a class of infinite-dimensional Lie algebras called the extended Schrödinger-Virasoro Lie algebra $\mathcal{L}$, which can be viewed as an extension of the Schrödinger-Virasoro Lie algebra by a conformal current of weight 1 and generated by $\{L_n, M_n, N_n, Y_p \,|\, n \in \mathbb{Z},\ p \in \mathbb{Z} + 1/2\}$ with the following Lie brackets:

$$\begin{array}{lll}
[L_m, L_n] = (n-m)L_{n+m}, & [L_m, N_n] = nN_{m+n}, & [L_m, M_n] = nM_{n+m}, \\
[L_n, Y_p] = (p-n/2)Y_{p+n}, & [N_m, Y_p] = Y_{m+p}, & [N_m, M_n] = 2M_{m+n}, \\
[M_n, Y_p] = [N_m, N_n] = 0, & [M_m, M_n] = 0, & [Y_p, Y_q] = (q-p)M_{p+q}.
\end{array} \tag{1.1}$$

Note that $\mathcal{L}$ is centerless and finitely generated with a generating set $\{L_{-2}, L_{-1}, L_1, L_2, N_1, Y_{1/2}\}$. Moreover, it is $\frac{1}{2}\mathbb{Z}$ -graded by

$$\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_{n/2} = \big(\bigoplus_{n\in Z} \mathcal{L}_n\big) \bigoplus \big(\bigoplus_{n\in\mathbb{Z}} \mathcal{L}_{n+1/2}\big),$$

where $\mathcal{L}_n = span\{L_n, M_n, N_n\}$ and $\mathcal{L}_{n+1/2} = span\{Y_{n+1/2}\}$, for all $n \in \mathbb{Z}$. The derivations, central extensions and automorphisms of $\mathcal{L}$ have been studied in [5].

To search for the solutions of the Yang-Baxter quantum equation, Drinfel'd [1] introduced the notion of Lie bialgebras in 1983. Since then, a number of people have studied further Lie bialgebra structures (e.g., [3, 11, 12, 14–16). Witt type Lie bialgebras introduced in [20] were classified in [17]. This work has been generalized in [19, 23]. Lie bialgebra structures on generalized Virasoro-like and

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Block Lie algebras were investigated in [11, 24]. Drinfel'd [2] posed the problem that whether or not there exists a general way to quantilize all Lie bialgebras. Etingof and Kazhdan [3] gave a positive answer to this problem, but there does not exist an uniform method to realize quantilizations of all Lie bialgebras. Actually, investigating Lie bialgebras and quantilizations is a complicated problem. The authors in [8] prove that not all Lie bialgebra structures on the Schrödinger-Virasoro Lie algebra are triangular coboundary. For the extended Schrödinger-Virasoro Lie algebra $\mathcal{L}$, this is not the case. Namely, we obtain that all Lie bialgebra structures on $\mathcal{L}$ are triangular coboundary. In particular, we derive that the first cohomology group $H^1(\mathcal{L}, \mathcal{L}\otimes\mathcal{L})$ is trivial.

§2. Preliminaries

Throughout the paper, $\mathbb{F}$ denotes a filed with characteristic zero. All vector spaces and tensor products are over $\mathbb{F}$. Let $\mathbb{Z}_+$ (resp. $\mathbb{Z}_{>0}$) be the set of all nonnegative (resp. positive) integers and $\mathbb{Z}^*$ be the set of all nonzero elements of $\mathbb{Z}$.

Let L be a vector space, ξ the *cyclic map* of $L\otimes L\otimes L$, namely, $\xi(x_1\otimes x_2\otimes x_3)=x_2\otimes x_3\otimes x_1$ for $x_1,x_2,x_3\in L$, and τ the *twist map* of $L\otimes L$, i.e., $\tau(x\otimes y)=y\otimes x$ for $x,y\in L$. A *Lie algebra* is a pair (L,δ), where $\delta: L\otimes L\to L$ is a bilinear map with the conditions:

$$\mathrm{Ker}(\,\mathrm{Id}-\tau)\subset \mathrm{Ker}\,\delta,\quad \delta\cdot(\,\mathrm{Id}\otimes\delta)\cdot(\,\mathrm{Id}+\xi+\xi^2)=0: L\otimes L\otimes L\to L,$$

where Id is the identity map. Dually, a *Lie coalgebra* is a pair (L,Δ) with a linear map $\Delta: L\to L\otimes L$ satisfying:

$$\mathrm{Im}\,\Delta\subset \mathrm{Im}(\,\mathrm{Id}-\tau),\quad (\,\mathrm{Id}+\xi+\xi^2)\cdot(\,\mathrm{Id}\otimes\Delta)\cdot\Delta=0: L\to L\otimes L\otimes L. \tag{2.1}$$

A *Lie bialgebra* is a triple (L,δ,Δ) such that (L,δ) is a Lie algebra, (L,Δ) is a Lie coalgebra, and the following compatible condition holds:

$$\Delta\delta(x\otimes y)=x\cdot\Delta y-y\cdot\Delta x,\quad \forall\ x,y\in L. \tag{2.2}$$

where "$\cdot$" means the *diagonal adjoint action*, i.e., $x\cdot(\sum_i a_i\otimes b_i)=\sum_i([x,a_i]\otimes b_i+a_i\otimes[x,b_i])$, and in general, $\delta(x\otimes y)=[x,y]$, for all $x,y,a_i,b_i\in L$.

Denote by $\mathcal{U}$ the universal enveloping algebra of L and 1 the identity element of $\mathcal{U}$. For any $r=\sum_i a_i\otimes b_i\in L\otimes L$, define $\boldsymbol{c}(r)\in\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}$ by

$$\boldsymbol{c}(r)=[r^{12},r^{13}]+[r^{12},r^{23}]+[r^{13},r^{23}],$$

where $r^{12}=\sum_i a_i\otimes b_i\otimes 1,\quad r^{13}=\sum_i a_i\otimes 1\otimes b_i,\quad r^{23}=\sum_i 1\otimes a_i\otimes b_i$. Obviously,

$$\boldsymbol{c}(r)=\sum_{i,j}[a_i,a_j]\otimes b_i\otimes b_j+\sum_{i,j}a_i\otimes[b_i,a_j]\otimes b_j+\sum_{i,j}a_i\otimes a_j\otimes[b_i,b_j].$$

Definition 2.1 (1) A *coboundary Lie bialgebra* (L,δ,Δ,r) is a Lie bialgebra such that the cobracket Δ is an inner derivation, i.e., there exists an element $r\in L\otimes L$ such that

$$\Delta(x)=x\cdot r\quad \text{for all}\quad x\in L. \tag{2.3}$$

Δ is called a coboundary of r, denoted by Δ_r.

(2) A coboundary Lie bialgebra (L, δ, Δ, r) is called *triangular* if it satisfies the following *classical Yang-Baxter Equation* (CYBE):

$$\boldsymbol{c}(r) = 0. \tag{2.4}$$

(3) An element $r \in \mathrm{Im}(\mathrm{Id} - \tau) \subset L \otimes L$ is said to satisfy the *modified Yang-Baxter equation* (MYBE) if

$$x \cdot \boldsymbol{c}(r) = 0, \ \ \forall\, x \in L. \tag{2.5}$$

The following famous results are due to Drinfel'd [2], Michaelis [14] and Taft [20], respectively. We combine them into one theorem as follows:

Theorem 2.2 (i) *For a Lie algebra $(L, [\cdot,\cdot])$ and $r \in \mathrm{Im}(\mathrm{Id} - \tau) \subset L \otimes L$, the triple $(L, [\cdot,\cdot], \Delta_r)$ is a Lie bialgebra if and only if r satisfies MYBE.*

(ii) *Let L be a Lie algebra containing two linear independent elements a, b satisfying $[a, b] = kb$ for some nonzero $k \in \mathbb{F}$, and set $r = a \otimes b - b \otimes a$. Then r is a solution of CYBE and equips L with a structure of triangular coboundary Lie bialgebra.*

(iii) *Let L be a Lie algebra and $r \in \mathrm{Im}(\mathrm{Id} - \tau) \subset L \otimes L$. Then for any $x \in L$,*

$$(\mathrm{Id} + \xi + \xi^2) \cdot (1 + \Delta_r) \cdot \Delta_r(x) = x \cdot c(r).$$

§3. Structures of Lie bialgebra of the extended Schrödinger-Virasoro Lie algebra

Regard $\mathcal{V} = \mathcal{L} \otimes \mathcal{L}$ as a $\mathcal{L}$-module under the adjoint diagonal action. A linear map $D : \mathcal{L} \to \mathcal{V}$ is called a *derivation* if

$$D([x, y]) = x \cdot D(y) - y \cdot D(x) \ \text{ for all } \ x,\ y\ \in \mathcal{L}. \tag{3.1}$$

If there exists some $v \in \mathcal{V}$ such that $D(x) = x \cdot v$, then D is called an *inner derivation*. Denote by v_{inn} the inner derivation determined by v. Let $\mathrm{Der}(\mathcal{L}, \mathcal{V})$ (resp. $\mathrm{Inn}(\mathcal{L}, \mathcal{V})$) be the set of all derivations (resp. inner derivations). Then it is well known that $H^1(\mathcal{L}, \mathcal{V}) \cong \mathrm{Der}(\mathcal{L}, \mathcal{V})/\mathrm{Inn}(\mathcal{L}, \mathcal{V})$, where $H^1(\mathcal{L}, \mathcal{V})$ is the *first cohomology group* of the Lie algebra $\mathcal{L}$ with coefficients in $\mathcal{V}$.

A derivation $D \in \mathrm{Der}(\mathcal{L}, \mathcal{V})$ is *homogeneous of degree* $\alpha \in \frac{1}{2}\mathbb{Z}$ if $D(\mathcal{L}_p) \subset \mathcal{V}_{\alpha+p}$ for all $p \in \frac{1}{2}\mathbb{Z}$. Let $\mathrm{Der}(\mathcal{L}, \mathcal{V})_\alpha$ be the set of all the homogeneous derivations of degree α. For any $D \in \mathrm{Der}(\mathcal{L}, \mathcal{V})$ and $\alpha \in \frac{1}{2}\mathbb{Z}$, define a linear map $D_\alpha : \mathcal{L} \to \mathcal{V}$ as follows: for any $\mu \in \mathcal{L}_q$ with $q \in \frac{1}{2}\mathbb{Z}$, write $D(\mu) = \sum_{p \in \frac{1}{2}\mathbb{Z}} \mu_p$ with $\mu_p \in \mathcal{V}_p$, then we set $D_\alpha(\mu) = \mu_{q+\alpha}$. Obviously, $D_\alpha \in \mathrm{Der}(\mathcal{L}, \mathcal{V})_\alpha$ and we have

$$D = \textstyle\sum_{\alpha \in \frac{1}{2}\mathbb{Z}} D_\alpha, \tag{3.2}$$

which holds in the sense that only finitely many $D_\alpha(u) \neq 0$ and $D(u) = \sum_{\alpha \in \frac{1}{2}\mathbb{Z}} D_\alpha(u)$ for any $u \in \mathcal{L}$. Actually, for any $D \in \mathrm{Der}(\mathcal{L}, \mathcal{V})$, (3.2) is a finite sum, referring to [8] for details.

Lemma 3.1 $H^1(\mathcal{L}_0, \mathcal{V}_{n/2}) = 0$ for all $n \in \mathbb{Z}^*$.

Proof. For any $D \in Der(\mathcal{L}, \mathcal{V})$, we have $D = \sum_{n\in\mathbb{Z}} D_{n/2}$. Suppose $n \neq 0$, then the restriction of $D_{n/2}$ to $\mathcal{L}_0$ induces a derivation from $\mathcal{L}_0$ to the $\mathcal{L}_0$-module $\mathcal{V}_{n/2}$. That is, $D_{n/2}|_{\mathcal{L}_0} \in \mathrm{Der}(\mathcal{L}_0, \mathcal{V}_{n/2})$. Conveniently, we denote $D_{n/2}|_{\mathcal{L}_0}$ by $D_{n/2}$. Let $r = \frac{2}{n} D_{n/2}(L_0) \in \mathcal{V}_{n/2}$. For any $X_0 \in \mathcal{L}_0$, one has $\frac{n}{2} D_{n/2}(X_0) = L_0 \cdot D_{n/2}(X_0) = X_0 \cdot D_{n/2}(L_0)$, since $[L_0, X_0] = 0$. It follows $D_{n/2}(X_0) = X_0 \cdot r$, which implies $D_{n/2}$ is inner. $\square$

Lemma 3.2 $\mathrm{Hom}_{\mathcal{L}_0}(\mathcal{V}_{\mathrm{m}/2}, \mathcal{V}_{\mathrm{n}/2}) = 0$ for all $m \neq n$.

Proof. Let $f \in \mathrm{Hom}_{\mathcal{L}_0}(\mathcal{V}_{\mathrm{m}/2}, \mathcal{V}_{\mathrm{n}/2}) = 0$ with $m \neq n$. One has $f([X_0, E_{m/2}]) = [X_0, f(E_{m/2})]$ for any $X_0 \in \mathcal{L}_0$ and $E_{m/2} \in \mathcal{V}_{m/2}$. In particular, $f([L_0, E_{m/2}]) = [L_0, f(E_{m/2})]$. That is, $\frac{m}{2} f(E_{m/2}) = \frac{n}{2} f(E_{m/2})$. It follows $f(E_{m/2}) = 0$, since $m \neq n$. Consequently, $f = 0$. $\square$

Taking these two Lemmas above into account, we can immediately derive the following result from Proposition 1.2 in [4].

Proposition 3.3 $\mathrm{Der}(\mathcal{L}, \mathcal{V}) = \mathrm{Der}_0(\mathcal{L}, \mathcal{V}) + \mathrm{Inn}(\mathcal{L}, \mathcal{V})$.

Lemma 3.4 Let $\mathcal{L}^{\otimes n} = \mathcal{L} \otimes \cdots \otimes \mathcal{L}$ be the tensor product of n copies of $\mathcal{L}$, and regard $\mathcal{L}^{\otimes n}$ as an $\mathcal{L}$-module under the adjoint diagonal action. Suppose $r \in \mathcal{L}^{\otimes n}$ satisfying $x \cdot r = 0$, $\forall\ x \in \mathcal{L}$. Then $r = 0$.

Proof. It is easy to see that $\mathcal{L}^{\otimes n}$ is $\frac{1}{2}\mathbb{Z}$-graded by

$$\mathcal{L}_p^{\otimes n} = \sum_{p_1+p_2+\cdots+p_n=p} \mathcal{L}_{p_1} \otimes \mathcal{L}_{p_2} \otimes \cdots \otimes \mathcal{L}_{p_n}, \quad \forall\ \ p,\ p_i \in \frac{1}{2}\mathbb{Z}, \quad i = 1, 2, \cdots n.$$

Write $r = \sum_{p\in\frac{1}{2}\mathbb{Z}} r_p$ as a finite sum with $r_p \in \mathcal{L}_p^{\otimes n}$. By hypothesis, $L_0 \cdot r = 0$, which implies $r = r_0$. So $r = \sum\limits_{r_1+r_2+\cdots+r_n=0} c_{r_1,r_2,\cdots r_n} E_{r_1} \otimes E_{r_2} \otimes \cdots E_{r_n}$ for some $c_{r_1,r_2,\cdots r_n} \in \mathbb{F}$ and $E_{r_i} \in \mathcal{L}_{r_i}$ with $r_i \in \frac{1}{2}\mathbb{Z}$. Since $M_0 \cdot r = 0$ by the assumption, all the coefficients of the terms containing N_j for $j \in \mathbb{Z}$ are zero, hence these terms in the sum vanish. Similarly, by $N_0 \cdot r = 0$, one can kill the coefficients of the terms containing M_j and $Y_{j+1/2}$ with $j \in \mathbb{Z}$. Now we can rewrite r by

$$r = \sum_{r_1+r_2+\cdots+r_n=0} c_{r_1,r_2,\cdots r_n} L_{r_1} \otimes L_{r_2} \otimes \cdots L_{r_n} \quad \text{for some} \quad c_{r_1,r_2,\cdots r_n} \in \mathbb{F}.$$

But $M_1 \cdot r = 0$ forces all the coefficients $c_{r_1,r_2,\cdots r_n}$ are zero. This proves the lemma. $\square$

Theorem 3.5 $\mathrm{Der}(\mathcal{L}, \mathcal{V}) = \mathrm{Inn}(\mathcal{L}, \mathcal{V})$.

Proof. It suffices to show $\mathrm{Der}_0(\mathcal{L}, \mathcal{V}) \subseteq \mathrm{Inn}(\mathcal{L}, \mathcal{V})$ by virtue of Proposition 3.3. For any $0 \neq D \in \mathrm{Der}_0(\mathcal{L}, \mathcal{V})$, we shall prove that the zero derivation is obtained after a number of steps in each of which D is replaced by $D - u_{\mathrm{inn}}$ for some $u \in \mathcal{V}_0$. This will be done by a little bit complicated calculations. For clarity, we divide them into three claims.

Claim 1. $D(L_0) = 0$.

In fact, for any $X_p \in \mathcal{L}$ with $p \in \frac{1}{2}\mathbb{Z}$, applying D to $[L_0, X_p] = pX_p$, one has $X_p \cdot D(L_0) = 0$. Then it follows from Lemma 3.4 that $D(L_0) = 0$.

Claim 2. $D(L_{\pm 1}) = 0$.

For any $n \in \mathbb{Z}$, one can write $D(L_n)$, $D(M_n)$, $D(N_n)$ and $D(Y_{n-1/2})$ as follows:

$$\begin{aligned}
D(L_n) = \sum_{i\in\mathbb{Z}}(&a_{1,i}^{(n)} L_i\otimes L_{n-i}+a_{2,i}^{(n)} L_i\otimes M_{n-i}+a_{3,i}^{(n)} M_i\otimes L_{n-i}+a_{4,i}^{(n)} L_i\otimes N_{n-i}+a_{5,i}^{(n)} N_i\otimes L_{n-i}\\
&+a_{6,i}^{(n)} M_i\otimes M_{n-i}+a_{7,i}^{(n)} M_i\otimes N_{n-i}+a_{8,i}^{(n)} N_i\otimes M_{n-i}+a_{9,i}^{(n)} N_i\otimes N_{n-i}+a_{10,i}^{(n)} Y_{i-1/2}\otimes Y_{n-i+1/2}),\\
D(M_n) = \sum_{i\in\mathbb{Z}}(&b_{1,i}^{(n)} L_i\otimes L_{n-i}+b_{2,i}^{(n)} L_i\otimes M_{n-i}+b_{3,i}^{(n)} M_i\otimes L_{n-i}+b_{4,i}^{(n)} L_i\otimes N_{n-i}+b_{5,i}^{(n)} N_i\otimes L_{n-i}\\
&+b_{6,i}^{(n)} M_i\otimes M_{n-i}+b_{7,i}^{(n)} M_i\otimes N_{n-i}+b_{8,i}^{(n)} N_i\otimes M_{n-i}+b_{9,i}^{(n)} N_i\otimes N_{n-i}+b_{10,i}^{(n)} Y_{i-1/2}\otimes Y_{n-i+1/2}),\\
D(N_n) = \sum_{i\in\mathbb{Z}}(&d_{1,i}^{(n)} L_i\otimes L_{n-i}+d_{2,i}^{(n)} L_i\otimes M_{n-i}+d_{3,i}^{(n)} M_i\otimes L_{n-i}+d_{4,i}^{(n)} L_i\otimes N_{n-i}+d_{5,i}^{(n)} N_i\otimes L_{n-i}\\
&+d_{6,i}^{(n)} M_i\otimes M_{n-i}+d_{7,i}^{(n)} M_i\otimes N_{n-i}+d_{8,i}^{(n)} N_i\otimes M_{n-i}+d_{9,i}^{(n)} N_i\otimes N_{n-i}+d_{10,i}^{(n)} Y_{i-1/2}\otimes Y_{n-i+1/2}),\\
D(Y_{n-1/2}) = \sum_{i\in\mathbb{Z}}(&\alpha_{n,i} L_i\otimes Y_{n-1/2-i}+\alpha_{n,i}^{\dagger} Y_{i-1/2}\otimes L_{n-i}+\beta_{n,i} M_i\otimes Y_{n-1/2-i}+\beta_{n,i}^{\dagger} Y_{i-1/2}\otimes M_{n-i}\\
&+\gamma_{n,i} N_i \otimes Y_{n-1/2-i} + \gamma_{n,i}^{\dagger} Y_{i-1/2} \otimes N_{n-i}).
\end{aligned}$$

Note that all the sums are finite. For any $n \in \mathbb{Z}$, one can easily get the following identities by (1.1):

$$\begin{aligned}
&L_1 \cdot (N_n \otimes N_{-n}) = nN_{n+1} \otimes N_{-n} - nN_n \otimes N_{1-n},\\
&L_1 \cdot (M_n \otimes N_{-n}) = nM_{n+1} \otimes N_{-n} - nM_n \otimes N_{1-n},\\
&L_1 \cdot (N_n \otimes M_{-n}) = nN_{n+1} \otimes M_{-n} - nN_n \otimes M_{1-n},\\
&L_1 \cdot (M_n \otimes M_{-n}) = nM_{n+1} \otimes M_{-n} - nM_n \otimes M_{1-n},\\
&L_1 \cdot (L_n \otimes N_{-n}) = (n-1)L_{n+1} \otimes N_{-n} - nL_n \otimes N_{1-n},\\
&L_1 \cdot (N_n \otimes L_{-n}) = nN_{n+1} \otimes L_{-n} - (1+n)N_n \otimes L_{1-n},\\
&L_1 \cdot (L_n \otimes M_{-n}) = (n-1)L_{n+1} \otimes M_{-n} - nL_n \otimes M_{1-n},\\
&L_1 \cdot (M_n \otimes L_{-n}) = nM_{n+1} \otimes L_{-n} - (1+n)M_n \otimes L_{1-n},\\
&L_1 \cdot (L_n \otimes L_{-n}) = (n-1)L_{n+1} \otimes L_{-n} - (1+n)L_n \otimes L_{1-n},\\
&L_1 \cdot (Y_{n-1/2} \otimes Y_{1/2-n}) = (n-1)Y_{n+1/2} \otimes Y_{1/2-n} - nY_{n-1/2} \otimes Y_{3/2-n}.
\end{aligned}$$

Let $Q_i = \max\{|\mathrm{p}| \mid \mathrm{a}_{\mathrm{i,p}}^{(1)} \neq 0\}$ for $i = 1,\cdots,10$. Applying $D - u_{\mathrm{inn}}$ to L_1, where u is a proper linear combination of $L_p \otimes L_{-p}$, $L_p \otimes M_{-p}$, $M_p \otimes L_{-p}$, $M_p \otimes M_{-p}$, $L_p \otimes N_{-p}$, $N_p \otimes L_{-p}$, $M_p \otimes N_{-p}$, $N_p \otimes M_{-p}$, $N_p \otimes N_{-p}$ and $Y_{p-1/2} \otimes Y_{1/2-p}$ with $p \in \mathbb{Z}$, and using induction on $\sum_{i=1}^{10} Q_i$, one can safely suppose

$$a_{1,i}^{(1)} = a_{6,j}^{(1)} = a_{7,j}^{(1)} = a_{8,j}^{(1)} = a_{9,j}^{(1)} = 0, \qquad \text{for } i \neq -1, 2, \;\; j \neq 0, 1, \tag{3.3}$$

$$a_{2,k}^{(1)} = a_{4,k}^{(1)} = a_{10,k}^{(1)} = a_{3,n}^{(1)} = a_{5,n}^{(1)} = 0, \quad \text{for } k \neq 0, 2, \;\; n \neq \pm 1. \tag{3.4}$$

Applying D to $[L_1, L_{-1}] = -2L_0$ and using $D(L_0) = 0$, we have

$$\begin{aligned}
&\sum_{p\in\mathbb{Z}}\Big(\big((p-2)a^{(-1)}_{1,p-1} - (p+2)a^{(-1)}_{1,p} + (p-2)a^{(1)}_{1,p} - (p+2)a^{(1)}_{1,p+1}\big)L_p\otimes L_{-p}\\
&+\big((p-2)a^{(-1)}_{2,p-1} - (p+1)a^{(-1)}_{2,p} + (p-1)a^{(1)}_{2,p} - (p+2)a^{(1)}_{2,p+1}\big)L_p\otimes M_{-p}\\
&+\big((p-1)a^{(-1)}_{3,p-1} - (p+2)a^{(-1)}_{3,p} + (p-2)a^{(1)}_{3,p} - (p+1)a^{(1)}_{3,p+1}\big)M_p\otimes L_{-p}\\
&+\big((p-2)a^{(-1)}_{4,p-1} - (p+1)a^{(-1)}_{4,p} + (p-1)a^{(1)}_{4,p} - (p+2)a^{(1)}_{4,p+1}\big)L_p\otimes N_{-p}\\
&+\big((p-1)a^{(-1)}_{5,p-1} - (p+2)a^{(-1)}_{5,p} + (p-2)a^{(1)}_{5,p} - (p+1)a^{(1)}_{5,p+1}\big)N_p\otimes L_{-p}\\
&+\big((p-1)a^{(-1)}_{6,p-1} - (p+1)a^{(-1)}_{6,p} + (p-1)a^{(1)}_{6,p} - (p+1)a^{(1)}_{6,p+1}\big)M_p\otimes M_{-p}\\
&+\big((p-1)a^{(-1)}_{7,p-1} - (p+1)a^{(-1)}_{7,p} + (p-1)a^{(1)}_{7,p} - (p+1)a^{(1)}_{7,p+1}\big)M_p\otimes N_{-p}\\
&+\big((p-1)a^{(-1)}_{8,p-1} - (p+1)a^{(-1)}_{8,p} + (p-1)a^{(1)}_{8,p} - (p+1)a^{(1)}_{8,p+1}\big)N_p\otimes M_{-p}\\
&+\big((p-1)a^{(-1)}_{9,p-1} - (p+1)a^{(-1)}_{9,p} + (p-1)a^{(1)}_{9,p} - (p+1)a^{(1)}_{9,p+1}\big)N_p\otimes N_{-p}\\
&+\big((p-2)a^{(-1)}_{10,p-1} - (p+1)a^{(-1)}_{10,p} + (p-2)a^{(1)}_{10,p} - (p+1)a^{(1)}_{10,p+1}\big)Y_{p-1/2}\otimes Y_{1/2-p}\Big) = 0.
\end{aligned}$$

In particular, one has

$$(p-2)a^{(-1)}_{1,p-1} - (p+2)a^{(-1)}_{1,p} + (p-2)a^{(1)}_{1,p} - (p+2)a^{(1)}_{1,p+1} = 0, \quad \forall\, p\in\mathbb{Z},$$

which together with the fact that $\{p\in\mathbb{Z}\,|\,a^{(-1)}_{1,p}\neq 0\}$ is finite and (3.3), forces

$$a^{(-1)}_{1,p} = 3a^{(-1)}_{1,-2} + a^{(-1)}_{1,-1} + 3a^{(1)}_{1,-1} = a^{(-1)}_{1,0} + 3a^{(-1)}_{1,1} + 3a^{(1)}_{1,2} = a^{(-1)}_{1,-1} + a^{(-1)}_{1,0} = 0, \tag{3.5}$$

for $p\in\mathbb{Z}\backslash\{-2,0,\pm1\}$. Similarly, comparing the coefficients of $L_p\otimes M_{-p}$, $M_p\otimes L_{-p}$,$L_p\otimes N_{-p}$, $N_p\otimes L_{-p}$, $M_p\otimes M_{-p}$, $M_p\otimes N_{-p}$, $N_p\otimes M_{-p}$, $N_p\otimes N_{-p}$ and $Y_{p-1/2}\otimes Y_{1/2-p}$ and taking (3.3) and (3.4) into account, one has

$$a^{(-1)}_{2,0} + 2a^{(-1)}_{2,1} = a^{(-1)}_{2,0} + 2a^{(-1)}_{2,-1} = a^{(-1)}_{3,-1} + 2a^{(-1)}_{3,0} = a^{(-1)}_{3,-1} + 2a^{(-1)}_{3,-2} = 0, \tag{3.6}$$

$$a^{(-1)}_{4,0} + 2a^{(-1)}_{4,1} = 2a^{(-1)}_{4,-1} + a^{(-1)}_{4,0} + a^{(1)}_{4,0} = a^{(-1)}_{5,-1} + 2a^{(-1)}_{5,0} = a^{(-1)}_{5,-1} + 2a^{(-1)}_{5,-2} = 0, \tag{3.7}$$

$$a^{(-1)}_{i,-1} + a^{(-1)}_{i,0} + a^{(1)}_{i,0} + a^{(1)}_{i,1} = 2a^{(-1)}_{10,-1} + a^{(-1)}_{10,0} + 2a^{(1)}_{10,0} = a^{(-1)}_{10,0} + 2a^{(-1)}_{10,1} + 2a^{(1)}_{10,2} = 0, \tag{3.8}$$

$$a^{(1)}_{2,p} = a^{(1)}_{3,p} = a^{(1)}_{4,p} = a^{(1)}_{5,p} = a^{(-1)}_{2,p_2} = a^{(-1)}_{3,p_3} = a^{(-1)}_{4,p_4} = a^{(-1)}_{5,p_5} = a^{(-1)}_{i,p_i} = a^{(-1)}_{10,p_{10}} = 0, \tag{3.9}$$

for any $p\in\mathbb{Z}$, $p_2\in\mathbb{Z}\backslash\{0,\pm1\}$, $p_3\in\mathbb{Z}\backslash\{0,-1,-2\}$, $p_4\in\mathbb{Z}\backslash\{0,\pm1\}$, $p_5\in\mathbb{Z}\backslash\{0,-1,-2\}$, $p_i\in\mathbb{Z}\backslash\{-1,0\}$ with $i=6,7,8,9$ and $p_{10}\in\mathbb{Z}\backslash\{0,\pm1\}$.

By (3.3) and (3.4) as well as applying D to $[L_1, N_0] = 0$ and $[L_1, M_0] = 0$, respectively, we have

$$a^{(1)}_{6,n} = a^{(1)}_{7,n} = a^{(1)}_{8,n} = a^{(1)}_{9,n} = a^{(1)}_{10,n} = 0, \quad \text{for all } n\in\mathbb{Z}, \tag{3.10}$$

$$d^{(0)}_{1,j_1} = d^{(0)}_{2,j_2} = d^{(0)}_{3,j_3} = d^{(0)}_{4,j_4} = d^{(0)}_{5,j_5} = d^{(0)}_{i,j} = d^{(0)}_{10,p} = 0, \tag{3.11}$$

$$b^{(0)}_{1,j_1} = b^{(0)}_{2,j_2} = b^{(0)}_{3,j_3} = b^{(0)}_{4,j_4} = b^{(0)}_{5,j_5} = b^{(0)}_{i,j} = b^{(0)}_{10,p} = 0, \tag{3.12}$$

$$d^{(0)}_{1,0} + 2d^{(0)}_{1,-1} = d^{(0)}_{1,0} + 2d^{(0)}_{1,1} = d^{(0)}_{l,0} + d^{(0)}_{l,1} = d^{(0)}_{k,0} + d^{(0)}_{k,-1} = 0, \tag{3.13}$$

$$b^{(0)}_{1,0} + 2b^{(0)}_{1,-1} = b^{(0)}_{1,0} + 2b^{(0)}_{1,1} = b^{(0)}_{l,0} + b^{(0)}_{l,1} = b^{(0)}_{k,0} + b^{(0)}_{k,-1} = 0, \tag{3.14}$$

where $j_1 \in \mathbb{Z}\backslash\{0,\pm 1\}$, $j_2 \in \mathbb{Z}\backslash\{0,1\}$, $j_3 \in \mathbb{Z}\backslash\{0,-1\}$, $j_4 \in \mathbb{Z}\backslash\{0,1\}$, $j_5 \in \mathbb{Z}\backslash\{0,-1\}$, $j \in \mathbb{Z}\backslash\{0\}$, $p \in \mathbb{Z}\backslash\{0,1\}$, $i = 6,\ 7,\ 8,\ 9$, $l = 2,\ 4,\ 10$ and $k = 3,\ 5$. Then it follows from (3.3), (3.9) and (3.10) that

$$D(L_1) \;=\; \textstyle\sum_{i\in\mathbb{Z}}\big(a^{(1)}_{1,i}L_i \otimes L_{1-i}\big) = a^{(1)}_{1,-1}L_{-1}\otimes L_2 + a^{(1)}_{1,2}L_2\otimes L_{-1}. \tag{3.15}$$

Furthermore, applying D to $[L_1, Y_{1/2}] = 0$, we get $D(L_1) = 0$ from (3.15). Similarly, applying D to $[L_{-1}, N_0] = 0$ and $[L_{-1}, M_0] = 0$, respectively, we obtain from (3.6)-(3.9) and (3.11)-(3.14) that

$$a^{(-1)}_{6,i} = a^{(-1)}_{7,i} = a^{(-1)}_{8,i} = a^{(-1)}_{9,i} = a^{(-1)}_{10,i} = b^{(0)}_{4,i} = b^{(0)}_{5,i} = d^{(0)}_{4,i} = d^{(0)}_{5,i} = 0, \tag{3.16}$$

$$d^{(0)}_{2,1} - a^{(-1)}_{2,0} = d^{(0)}_{3,-1} - a^{(-1)}_{3,-1} = b^{(0)}_{2,1} + a^{(-1)}_{4,0} = b^{(0)}_{3,-1} + a^{(-1)}_{5,0} = 0, \tag{3.17}$$

for all $i \in \mathbb{Z}$. Set $u := L_1 \otimes M_{-1} - L_0 \otimes M_0$. Observe that $L_1 \cdot u = 0$. Substitute $D + a^{(-1)}_{2,1}u_{inn}$ into the expression of $D(L_{-1})$, one can safely assume $a^{(-1)}_{2,1} = 0$, since such replacement would not affect the expression of $D(L_1)$. Similarly, set $u^{(1)} := M_{-1}\otimes L_1 - M_0 \otimes L_0$, $u^{(2)} := L_1 \otimes N_{-1} - L_0 \otimes N_0$, and $u^{(3)} := N_{-1}\otimes L_1 - N_0 \otimes L_0$, then replace D by $D + a^{(-1)}_{3,0}u^{(1)}_{inn}$, $D + a^{(-1)}_{4,1}u^{(2)}_{inn}$ and $D + a^{(-1)}_{5,0}u^{(3)}_{inn}$ in turn, one can assume $a^{(-1)}_{3,0} = a^{(-1)}_{4,1} = a^{(-1)}_{5,0} = 0$. Hence we get

$$D(L_{-1}) = a^{(-1)}_{1,-2}L_{-2}\otimes L_1 + a^{(-1)}_{1,-1}L_{-1}\otimes L_0 + a^{(-1)}_{1,0}L_0 \otimes L_{-1} + a^{(-1)}_{1,1}L_{-1}\otimes L_2,$$

by (3.5)-(3.7). Finally, using $D([L_{-1}, Y_{-1/2}]) = 0$ and (3.5), one has $D(L_{-1}) = 0$.

Claim 3. $D(L_{\pm 2}) = D(N_1) = D(Y_{1/2}) = 0$.

It follows (3.14) and (3.16) that

$$\begin{aligned} D(M_0) = {}& b^{(0)}_{1,-1}L_{-1}\otimes L_1 + b^{(0)}_{1,0}L_0\otimes L_0 + b^{(0)}_{1,1}L_1\otimes L_{-1} + b^{(0)}_{6,0}M_0\otimes M_0 + b^{(0)}_{7,0}M_0\otimes N_0 \\ & + b^{(0)}_{8,0}N_0\otimes M_0 + b^{(0)}_{9,0}N_0\otimes N_0 + b^{(0)}_{10,0}Y_{-1/2}\otimes Y_{1/2} + b^{(0)}_{10,1}Y_{1/2}\otimes Y_{-1/2}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} D(N_0) = {}& d^{(0)}_{1,-1}L_{-1}\otimes L_1 + d^{(0)}_{1,0}L_0\otimes L_0 + d^{(0)}_{1,1}L_1\otimes L_{-1} + d^{(0)}_{6,0}M_0\otimes M_0 + d^{(0)}_{7,0}M_0\otimes N_0 \\ & + d^{(0)}_{8,0}N_0\otimes M_0 + b^{(0)}_{9,0}N_0\otimes N_0 + d^{(0)}_{10,0}Y_{-1/2}\otimes Y_{1/2} + d^{(0)}_{10,1}Y_{1/2}\otimes Y_{-1/2}. \end{aligned} \tag{3.19}$$

Set $v^{(1)} := M_0 \otimes M_0$, $v^{(2)} := M_0 \otimes N_0$, $v^{(3)} := N_0 \otimes M_0$ and $v^{(4)} := Y_{1/2}\otimes Y_{-1/2} - Y_{-1/2}\otimes Y_{1/2}$. Observe that $L_{\pm 1}\cdot v^{(i)} = 0$, but $N_0 \cdot v^{(i)} \neq 0$ for $i = 1, 2, 3, 4$. Replacing D by $D - \frac{1}{4}d^{(0)}_{6,0}v^{(1)}_{inn}$, $D - \frac{1}{2}d^{(0)}_{7,0}v^{(2)}_{inn}$, $D - \frac{1}{2}d^{(0)}_{8,0}v^{(3)}_{inn}$ and $D - \frac{1}{2}d^{(0)}_{10,0}v^{(4)}_{inn}$ in turn in (3.19), one can assume that $d^{(0)}_{6,0} = d^{(0)}_{7,0} = d^{(0)}_{8,0} = d^{(0)}_{10,0} = 0$. By applying D to $[N_0, N_1] = 0$ and using (3.13), we have $d^{(0)}_{1,-1} = d^{(0)}_{1,1} = d^{(0)}_{1,-1} = d^{(0)}_{10,1} = 0$. Then it follows from $D([N_0, M_0]) = 2D(M_0)$ and (3.18)-(3.19)that $D(N_0) = 0$ and

$$D(M_0) = b^{(0)}_{7,0}M_0\otimes N_0 + b^{(0)}_{8,0}N_0\otimes M_0 + b^{(0)}_{10,0}Y_{-1/2}\otimes Y_{1/2} + b^{(0)}_{10,1}Y_{1/2}\otimes Y_{-1/2}\,. \tag{3.20}$$

Now considering $D([L_{\pm2},M_0])=0$ and $D([L_{\pm2},N_0])=0$, one has

$$D(L_2)=\textstyle\sum_i a_{1,i}^{(2)}L_i\otimes L_{2-i},\quad \text{and}\quad D(L_{-2})=\textstyle\sum_i a_{1,i}^{(-2)}L_i\otimes L_{-2-i}. \tag{3.21}$$

As a by-product, we also get $b_{10,0}^{(0)}=b_{10,1}^{(0)}=0$. Replacing D by $D+\frac{1}{2}b_{8,0}^{(0)}(N_0\otimes N_0)_{inn}$ in (3.20), one can assume $b_{8,0}^{(0)}=0$. Now we get from (3.20) that

$$D(M_0)=b_{7,0}^{(0)}M_0\otimes N_0. \tag{3.22}$$

Applying D to $[L_1,L_{-2}]=-3L_{-1}$ and $[L_{-1},L_2]=3L_1$, respectively, and using $D(L_{\pm1})=0$, we have

$$a_{1,p}^{(2)}=a_{1,q}^{(-2)}=2a_{1,1}^{(2)}+3a_{1,0}^{(2)}=a_{1,0}^{(2)}+4a_{1,-1}^{(2)}=a_{1,1}^{(2)}+a_{1,2}^{(2)}=a_{1,2}^{(2)}+4a_{1,3}^{(2)}=0, \tag{3.23}$$

$$2a_{1,-1}^{(-2)}+3a_{1,0}^{(-2)}=a_{1,0}^{(-2)}+4a_{1,1}^{(-2)}=a_{1,-1}^{(-2)}+a_{1,-2}^{(-2)}=a_{1,-2}^{(-2)}+4a_{1,-3}^{(-2)}=0, \tag{3.24}$$

where $p\in\mathbb{Z}\backslash\{0,\pm1,2,3\}$ and $q\in\mathbb{Z}\backslash\{-3,-2,0,\pm1\}$. Set $v:=L_{-1}\otimes L_1-2L_0\otimes L_0+L_1\otimes L_{-1}$ and take $D-\frac{1}{4}a_{1,0}^{(2)}v_{inn}$ in place of D in the first equation of (3.21), one can assume $a_{1,0}^{(2)}=0$. Then it follows (3.21) and (3.23) that $D(L_2)=0$. Consequently, one can easily get $D(L_{-2})=0$ by applying D to $[L_{-2},L_2]=4L_0$ and using (3.24).

Applying D to $[M_0,Y_{1/2}]=0$ and $[M_0,Y_{-1/2}]=0$, respectively, one has

$$\gamma_{1,i}^{\dagger}=\gamma_{0,i}^{\dagger}=\gamma_{0,j}=\gamma_{0,j}=0,\ \forall\ i\in\mathbb{Z},\ j\in\mathbb{Z}\backslash\{0\}. \tag{3.25}$$

Similarly, when D is applied to $[N_0,Y_{1/2}]=Y_{1/2}$ and $[N_0,Y_{-1/2}]=Y_{-1/2}$, respectively, it follows

$$\beta_{1,i}=\beta_{1,i}^{\dagger}=\beta_{0,i}=\beta_{0,i}^{\dagger}=0,\quad \text{for all}\quad i\in\mathbb{Z}.$$

Using $D([L_1,Y_{1/2}])=0$, $D([L_{-1},Y_{-1/2}])=0$ and $D(L_{\pm1})=0$, we have

$$\alpha_{1,i}=\alpha_{1,i}^{\dagger}=\alpha_{0,i}^{\dagger}=\alpha_{0,j}=0,\ \forall\ i\in\mathbb{Z}\backslash\{0,1\},j\in\mathbb{Z}\backslash\{0,-1\}, \tag{3.26}$$

$$\alpha_{1,0}+\alpha_{1,1}=\alpha_{0,0}+\alpha_{0,-1}=\alpha_{1,0}^{\dagger}+\alpha_{1,1}^{\dagger}=\alpha_{0,0}^{\dagger}+\alpha_{0,1}^{\dagger}=0. \tag{3.27}$$

Applying D to $[L_1,Y_{-1/2}]=-Y_{1/2}$, one has $\alpha_{0,0}=\alpha_{1,1}$, $\alpha_{0,0}^{\dagger}=\alpha_{1,0}^{\dagger}$ and $\gamma_{0,0}=\gamma_{1,0}$. Let $a=\alpha_{0,0}$, $b=\alpha_{0,0}^{\dagger}$ and $c=\gamma_{0,0}$. Hence, we can rewrite $D(Y_{\pm1/2})$ as follows:

$$D(Y_{1/2})=-aL_0\otimes Y_{1/2}+aL_1\otimes Y_{-1/2}+bY_{-1/2}\otimes L_1-bY_{1/2}\otimes L_0+cN_0\otimes Y_{1/2}, \tag{3.28}$$

$$D(Y_{-1/2})=aL_0\otimes Y_{-1/2}-aL_{-1}\otimes Y_{1/2}+bY_{-1/2}\otimes L_0-bY_{1/2}\otimes L_{-1}+cN_0\otimes Y_{-1/2}. \tag{3.29}$$

When D is applied to $[L_2,Y_{-1/2}]=-\frac{3}{2}Y_{3/2}$ and $[L_{-2},Y_{3/2}]=\frac{5}{2}Y_{-1/2}$, respectively, one has $a=b=0$, since $D(L_{\pm2})=0$. It follows $c=d_{7,0}^{(0)}=0$ by applying D to $[Y_{-1/2},Y_{1/2}]=M_0$, which proves $D(Y_{\pm1/2})=D(M_0)=0$ by (3.22), (3.28) and (3.29).

Now it is left to calculate $D(N_1)$. Firstly, using $D([N_0, N_1]) = 0$ and $D(N_0) = 0$, we have

$$d^{(1)}_{2,i} = d^{(1)}_{3,i} = d^{(1)}_{6,i} = d^{(1)}_{7,i} = d^{(1)}_{8,i} = d^{(1)}_{10,i} = 0, \quad \forall \;\; i \in \mathbb{Z}. \tag{3.30}$$

Then applying D to $[L_{-1}, N_1] = N_0$ and using $D(N_0) = 0$, we obtain

$$\begin{aligned} &d^{(1)}_{1,i_1} = d^{(1)}_{4,i_2} = d^{(1)}_{5,i_3} = d^{(1)}_{9,i_4} = d^{(1)}_{1,0} + d^{(1)}_{1,1} = d^{(1)}_{1,1} + 3d^{(1)}_{1,2} = d^{(1)}_{1,0} + 3d^{(1)}_{1,-1} = 0,\\ &d^{(1)}_{4,0} + 2d^{(1)}_{4,1} = d^{(1)}_{4,0} + 2d^{(1)}_{4,-1} = d^{(1)}_{5,1} + 2d^{(1)}_{5,0} = d^{(1)}_{5,1} + 2d^{(1)}_{5,2} = d^{(1)}_{9,0} + d^{(1)}_{9,1} = 0, \end{aligned} \tag{3.31}$$

where $i_1 \in \mathbb{Z}\backslash\{0, \pm 1, 2\}$, $i_2 \in \mathbb{Z}\backslash\{0, \pm 1\}$, $i_3 \in \mathbb{Z}\backslash\{0, 1, 2\}$, and $i_4 \in \mathbb{Z}\backslash\{0, 1\}$. Finally, by applying D to $[N_1, Y_{-1/2}] = Y_{1/2}$ and (3.31) as well as $D(Y_{\pm 1/2}) = 0$, we get

$$d^{(1)}_{1,i} = d^{(1)}_{4,i} = d^{(1)}_{5,i} = d^{(1)}_{9,i} = 0, \qquad \forall \;\; i \in \mathbb{Z},$$

which together with (3.30), yields $D(N_1) = 0$. Hence, the claim is proved, so is the theorem, since $\mathcal{L}$ is generated by $L_{\pm 1}$, $L_{\pm 2}$, N_1 and $Y_{1/2}$. $\square$

The following lemma is very useful to the main theorem in the paper.

Lemma 3.6 Suppose $v \in \mathcal{V}$ such that $x \cdot v \in \mathrm{Im}(\mathrm{Id} - \tau)$ for all $x \in \mathcal{L}$. Then $v \in \mathrm{Im}(\mathrm{Id} - \tau)$.

Proof. First note that $\mathcal{L} \cdot \mathrm{Im}(\mathrm{Id} - \tau) \subset \mathrm{Im}(\mathrm{Id} - \tau)$. We shall show that after several steps in each of which v is replaced by $v - u$ for some $u \in \mathrm{Im}(\mathrm{Id} - \tau)$, the zero element is obtained, which leads us to the result. Write $v = \sum_{n \in \frac{1}{2}\mathbb{Z}} v_n$. Obviously,

$$v \in \mathrm{Im}(\mathrm{Id} - \tau) \iff v_n \in \mathrm{Im}(\mathrm{Id} - \tau), \;\; \forall \; n \in \frac{1}{2}\mathbb{Z}. \tag{3.32}$$

Then $\sum_{n \in \frac{1}{2}\mathbb{Z}} n v_n = L_0 \cdot v \in \mathrm{Im}(\mathrm{Id} - \tau)$. By (3.32), $n v_n \in \mathrm{Im}(\mathrm{Id} - \tau)$. In particular, $v_n \in \mathrm{Im}(\mathrm{Id} - \tau)$ if $n \neq 0$. Thus when replacing v by $v - \sum_{n \in \frac{1}{2}\mathbb{Z}^*} v_n$, one can suppose $v = v_0 \in \mathcal{V}_0$. Write

$$\begin{aligned} v \;=\; &\sum_{i \in \mathbb{Z}} (a_i L_i \otimes L_{-i} + b_i L_i \otimes M_{-i} + c_i M_i \otimes L_{-i} + d_i M_i \otimes M_{-i} + e_i Y_{i-1/2} \otimes Y_{1/2-i}\\ &+\; f_i L_i \otimes N_{-i} + g_i N_i \otimes L_{-i} + h_i N_i \otimes N_{-i} + k_i M_i \otimes N_{-i} + r_i N_i \otimes M_{-i}). \end{aligned}$$

Since all the elements of the forms $E_i \otimes F_{-i} - F_{-i} \otimes E_i$ and $Y_{i-1/2} \otimes Y_{1/2-i} - Y_{1/2-i} \otimes Y_{i-1/2}$ are contained in $\mathrm{Im}(\mathrm{Id} - \tau)$, where $\{E_i, F_i\} \subset \{L_i, M_i, N_i\}$ for all $i \in \mathbb{Z}$. Replacing v by $v - u$, where u is a combination of some of these elements, we can assume

$$c_i = g_i = r_i = 0, \; \forall \;\; i \in \mathbb{Z}; \;\;\; a_i, \; d_i, h_i \neq 0 \implies \;\; i > 0 \;\; \text{or} \;\; i = 0; \;\;\; e_i \neq 0 \implies \;\; i > 0. \tag{3.33}$$

Then v can be rewritten as

$$\begin{aligned} v \;\;=\;\; &\sum_{i \in \mathbb{Z}_+} (a_i L_i \otimes L_{-i} + d_i M_i \otimes M_{-i} + h_i N_i \otimes N_{-i})\\ &+\; \sum_{i \in \mathbb{Z}} (b_i L_i \otimes M_{-i} + f_i L_i \otimes N_{-i} + k_i M_i \otimes N_{-i}) + \sum_{i \in \mathbb{Z}_{>0}} e_i Y_{i-1/2} \otimes Y_{1/2-i}. \end{aligned} \tag{3.34}$$

Assume $a_p \neq 0$ for some $p > 0$. Choose $q > 0$ such that $q \neq p$. Then $L_{p+q} \otimes L_{-p}$ appears in $L_q \cdot v$, but (3.33) implies the term $L_{-p} \otimes L_{p+q}$ does not appear in $L_q \cdot v$, which contradicts the fact that $L_q \cdot v \in \mathrm{Im}(\mathrm{Id} - \tau)$. Hence we get $a_i = 0,\ \forall\ i \in \mathbb{Z}^*$. Similarly, one can suppose $d_i = h_i = 0,\ \forall\ i \in \mathbb{Z}^*$ and $e_i = 0,\ \forall\ i \in \mathbb{Z}$. Then (3.34) becomes

$$v = \sum_{i \in \mathbb{Z}} (b_i L_i \otimes M_{-i} + f_i L_i \otimes N_{-i} + k_i M_i \otimes N_{-i}) + a_0 L_0 \otimes L_0 + d_0 M_0 \otimes M_0 + h_0 N_0 \otimes N_0 \,. \quad (3.35)$$

By $\mathrm{Im}(\mathrm{Id} - \tau) \subset \mathrm{Ker}(\mathrm{Id} + \tau)$ and our hypothesis $\mathcal{L} \cdot v \subset \mathrm{Im}(\mathrm{Id} - \tau)$, we have

$$0 = (\mathrm{Id} + \tau) M_0 \cdot v = -4h_0(M_0 \otimes N_0 + N_0 \otimes M_0) - 2\sum_{i \in \mathbb{Z}} f_i (L_i \otimes M_{-i} + M_{-i} \otimes L_i) - 4\sum_{i \in \mathbb{Z}} k_i (M_i \otimes M_{-i}) \,.$$

Comparing the coefficients, one gets $h_0 = 0$ and $f_i = k_i = 0,\ \forall\ i \in \mathbb{Z}$. Similarly, $(\mathrm{Id} + \tau)(N_0 \cdot v) = 0$ implies $d_0 = 0$ and $b_i = 0$ for all $i \in \mathbb{Z}$; and $(\mathrm{Id} + \tau)(L_1 \cdot v) = 0$ leads to $a_0 = 0$. Then the lemma follows from (3.35). □

By now we have enough in hand to classify the Lie bialgebra structures on the extended Schrödinger-Virasoro Lie algebra. The following theorem is the central result of the paper.

Theorem 3.7 *Let $(\mathcal{L}, [\cdot,\cdot])$ be the extended Schrödinger-Virasoro Lie algebra. Then each Lie bialgebra structure on $\mathcal{L}$ is triangular coboundary.*

Proof. Let $(\mathcal{L}, [\cdot,\cdot], \Delta)$ be a Lie bialgebra structure on $\mathcal{L}$. Thanks to Theorem 3.5, there exists some $r \in \mathcal{V}$ such that $\Delta = \Delta_r$. By (2.1), $\mathrm{Im}(\Delta) \subset \mathrm{Im}(\mathrm{Id} - \tau)$. Then it follows from Lemma 3.6 that $r \in \mathrm{Im}(\mathrm{Id} - \tau)$. But (2.1), Theorem 2.2(iii) and Lemma 3.4 show that $\boldsymbol{c}(r) = 0$; as a result, $(\mathcal{L}, [\cdot,\cdot], \Delta)$ is a triangular coboundary Lie bialgebra by Definition 2.1. □

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